South East Asian J. of Mathematics and Mathematical Sciences
Vol. 21, No. 3 (2025), pp. 239-256

DOI: 10.56827/SEAJMMS.2025.2103.15 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

PARTITION DIMENSION OF BENZENOID SYSTEMS AND HEXAGONAL CACTUS CHAINS

R. Nithya Raj, Mana Donganont*, M. Chandrakala and V. Govindan

Department of Mathematics and Chemistry,
Hindustan Institute of Technology and Science,
Chennai - 603103, INDIA

E-mail : nithyarajmaths@gmail.com, chandrakala.mannar@gmail.com, govindoviya@gmail.com

*School of Science,
University of Phayao, Phayao 56000, THAILAND

E-mail : mana.do@up.ac.th

(**Received:** *Aug. 01, 2025* **Accepted:** *Dec. 25, 2025* **Published:** *Dec. 30, 2025)*

Abstract: This paper determines the partition dimension of selected hexagonal graph structures, namely pericondensed benzenoid systems R_n, B_n, and hexagonal cactus chains PC_n, OC_n, and MC_n. Using combinatorial techniques, we obtain exact partition dimensions for these classes. Our findings provide valuable insights into their structural properties and have applications in areas such as molecular chemistry and computational biology.

Keywords and Phrases: Pericondensed benzenoid system, hexagonal catus chains, partition resolving set, partition dimension.

2020 Mathematics Subject Classification: 05C12, 05C90, 05C15, 05C62.

1. Introduction

Chemical graph structures such as pericondensed benzenoid systems and hexagonal cactus chains are used to model aromatic compounds. In these graphs, vertices represent atoms and edges represent chemical bonds. Pericondensed benzenoid systems form hexagonal grids corresponding to polycyclic aromatic hydrocarbons,

while hexagonal cactus chains, including para-, ortho-, and meta-chains, represent different hexagonal ring connections. Studying these graphs supports the analysis of molecular structure and related properties.

Gutman et al., [10], studied the relations for calculating the Hosoya indices of two specific benzenoid systems, R_n and B_n. Oz and Cangul, [19], studied the Hosoya and Merrifield-Simmons indices of the benzenoid systems R_n and B_n by introducing four vectors for each system and its corresponding index. Ali et al., [21], discussed the Hosoya polynomial of hexagonal cactus chains. The topological characteristics of certain molecular cactus chain networks and their subdivisions were examined by Turaci and Durgut, [25], using line operators. Zahra et al., [27], studied the chain of hexagonal cacti extremal with respect to the eccentric connectivity index, Doslic and Maloy, [7], calculated the matchings and independent sets for hexagonal cactus chain.

Finding the metric dimension of networks along with its generalized version known as the partition dimension involves finding the smallest number of vertices or subsets required to uniquely identify the position of any other vertex based on distances within the network. Slater, [23], and Harary et al., [12], independently established the idea of a graph's metric dimension which become a very widely studied area in graph theory. A generalization of the metric dimension invented by Chartrand et al., [24], is the partition dimension of a graph. Ioan Tomescu, [4], gave the discrepancies between metric dimension and partition dimension of a connected graph. The partition dimension (PD) is based on the distances between sets of vertices and the metric dimension (MD) is based on the distances between vertices. Since it is NP-complete to determine the metric dimension of any graph, [8], the partition dimension problem is NP-complete as well.

Applications for the metric dimension can be found in many different fields and it has been well studied. Because of its adaptability and significance, researchers have used it in a variety of applications. For example, the similarity between several drugs has been assessed using the metric dimension, [15]. Metamaterials which are essential to the pharmaceutical business are intentionally created materials and structures, [26]. Further applications include the solution of issues related to pharmaceutical chemistry, [5], robot navigation assistance, [17], and combinatorial optimization problems, [1].

Partition dimension has various applications in chemistry and numerous studies have explored them in different chemical systems. The partition dimension of some wheel related graphs are studied in [14], one pentagonal Carbon nanocone structure's partition dimension bounds are found in [18] and the carbon nanocones' edge partition dimensions are found in [22]. In [20], the partition dimension of a

path graph is computed. There is a discussion of the partition resolvability of nanosheets and nanotubes made from an octagonal grid in [16]. The resolvability parameters of several chemical systems and other calculations with benzenoid chains are covered in [2, 3, 6, 9, 11, 13].

In this paper, in Section 2, basis definitions are discussed. In Section 3, we deal with the partition dimension of two special types of benzenoid systems and hexagonal cactus chains. Finally, theoretical importance and practical use of the constant partition dimension of some pericondensed benzenoid systems and hexagonal cactus chains together with some concluding remarks are discussed in Sections 4 and 5, respectively.

2. Basic concepts

In the present paper, a finite simple connected graph $G = (V, E)$ is considered where V and E are the sets of vertices and edges of graph G, respectively. The distance between two vertices α and β in a graph G denoted by $d(\alpha, \beta)$ is defined as the length of a shortest path between α and β, where the length is the number of edges in the path. If there is no path between α and β, the distance is considered infinite. The metric dimension of a graph G, denoted as $dim(G)$, is the smallest cardinality of a subset $S \subseteq V(G)$ such that for any two distinct vertices $u, v \in V(G)$, there exists a vertex $w \in S$ for which the distances $d(u, w)$ and $d(v, w)$ are different. This means that the set S uniquely identifies each vertex in the graph based on its distance to the vertices in S.

The partition dimension of a graph $G = (V, E)$ is the minimum number of subsets in a vertex partition $P = \{P_1, P_2, P_3, \ldots, P_k\}$ of V such that the collection of distance vectors $\{d(v, P_1), d(v, P_2), d(v, P_3), \ldots, d(v, P_k)\}$ uniquely identifies each vertex $v \in V$. Here $d(v, P_i)$ represents the minimum distance from the vertex v to any vertex in the subset P_i. The partition dimension of a graph G is denoted by $pd(G)$.

The partition dimension of a graph can be demonstrated by the following illustration: Consider the graph G in Fig. 1. Divide the set of vertices of G into the following four sets, S_1, S_2, S_3 and S_4, where $S_1 = \{t_9\}$, $S_2 = \{t_3\}$, $S_3 = \{t_6\}$, and $S_4 = \{t_1, t_2, t_4, t_5, t_7, t_8, t_{10}\}$. If $\Pi_1 = \{S_1, S_2, S_3, S_4\}$ is a resolving partition of G, then the identification of vertices with respect to Π_1 are $r(t_1|\Pi_1) = (3, 2, 3, 0)$, $r(t_2|\Pi_1) = (4, 3, 2, 0)$, $r(t_3|\Pi_1) = (3, 0, 3, 1)$, $r(t_4|\Pi_1) = (2, 1, 2, 0)$ $r(t_5|\Pi_1) = (3, 2, 1, 0)$, $r(t_6|\Pi_1) = (4, 3, 0, 1)$, $r(t_7|\Pi_1) = (1, 2, 3, 0)$, $r(t_8|\Pi_1) = (2, 3, 2, 0)$, $r(t_9|\Pi_1) = (0, 3, 4, 1)$, $r(t_{10}|\Pi_1) = (3, 4, 3, 0)$. It is observed that every vertex in G has a unique identification with respect to Π_1. Therefore, the resolving partition dimension of G is Π_1.

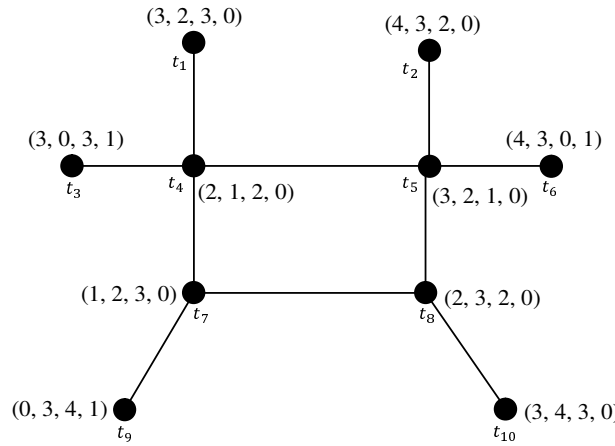

Figure 1: A graph G with resolving partition set of 4 parameters

For the same graph, a resolving partition of smaller cardinality is exhibited in Fig. 2 by $\Pi_2 = \{S_1, S_2, S_3\}$ where $S_1 = \{t_9\}$, $S_2 = \{t_1, t_2\}$, $S_3 = \{V(G) \setminus S_1 \cup S_2\}$. The representations of vertices in relation to Π_2 are $r(t_1|\Pi_2) = (3, 0, 1)$, $r(t_2|\Pi_2) = (4, 0, 1)$, $r(t_3|\Pi_2) = (3, 2, 0)$, $r(t_4|\Pi_2) = (2, 1, 0)$, $r(t_5|\Pi_2) = (3, 1, 0)$, $r(t_6|\Pi_2) = (4, 2, 0)$, $r(t_7|\Pi_2) = (1, 2, 0)$, $r(t_8 4|\Pi_2) = (2, 2, 0)$, $r(t_9|\Pi_2) = (0, 3, 1)$, $r(t_{10}|\Pi_2) = (3, 3, 0)$.

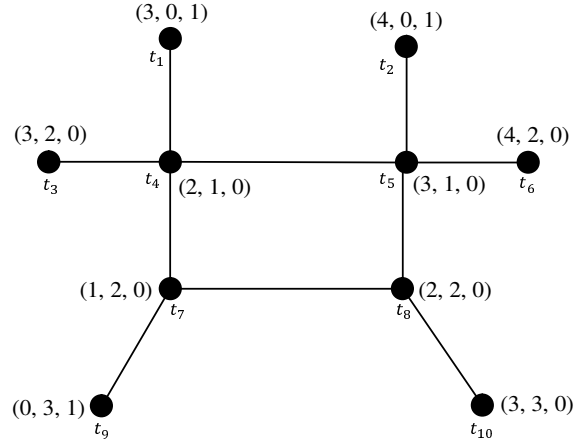

Figure 2: A graph G with resolving partition set of 3 parameters

The following theorems are recalled as we shall be needing them in the remaining parts of this work:

Theorem 1. [4] *If G is a non-trivial connected chemical network, then the partition dimension of G satisfies the inequality $pd(G) \leq dim(G) + 1$.*

Theorem 2. [4] *For each graph G, let Π be its resolving partition, and let $\alpha, \beta \in V(G)$. We can determine that α and β are in different classes of Π if $d(\alpha, \gamma) = d(\beta, \gamma)$ for any vertices $\gamma \in V(G) \setminus (\alpha, \beta)$.*

Theorem 3. [4] *Let G be a simple connected graph of order n. Consequently,*
(a) G is a path graph if and only if $pd(G)$ equals 2.
(b) G is a complete graph if and only if $pd(G)$ is n.

3. Main results

In this study, we determine the partition dimensions of two special benzenoid systems denoted by R_n and B_n along with the hexagonal cactus chains. By exploring the structural properties and resolving partition set of these graph families, we provide a detailed analysis of their partition dimensions. This result contributes to a deeper understanding of the graph identification process and has implications for both theoretical studies in graph theory and practical applications in chemical structure modeling and network design.

3.1. Two special Benzenoid systems R_n and B_n

A catacondensed benzenoid system is a type of polycyclic aromatic hydrocarbon (PAH) composed of hexagonal rings (benzene units) where no single hexagonal ring is shared by more than two other hexagons. Unlike pericondensed benzenoid systems where multiple rings can overlap, in catacondensed systems, the hexagons are arranged in such a way that the structure does not form any internal circuits, resembling more of a tree-like or chain-like configuration. These structures are often simpler in terms of connectivity compared to their pericondensed counterparts. Due to the restricted way in which the hexagonal units are connected, catacondensed benzenoid systems typically exhibit simpler topological properties, which can influence their chemical and physical characteristics. They are frequently studied in the context of chemical graph theory due to their well-defined boundaries and predictable growth patterns. Examples include linear polyacenes and other benzenoid chains that extend without forming closed loops within the system. The references [11, 15, 17, 26] contain some research on benzenoid (hexagonal) systems. Two categories of pericondensed benzenoid systems are represented in Fig.s 3 and 4 by R_n and B_n :

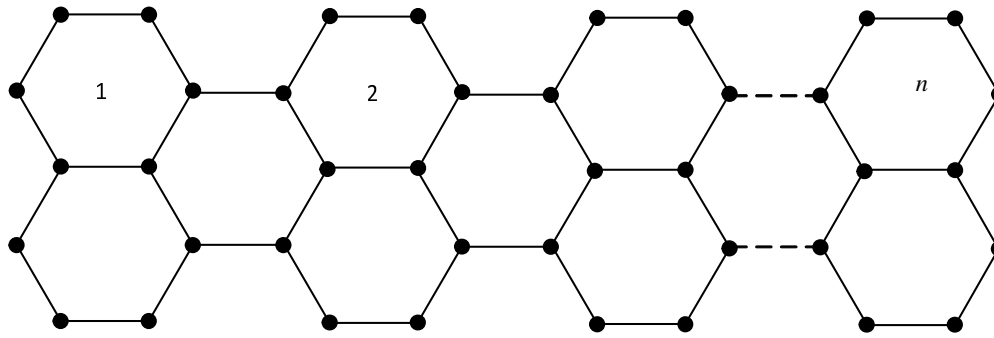

Figure 3: Benzoined system R_n of dimension n

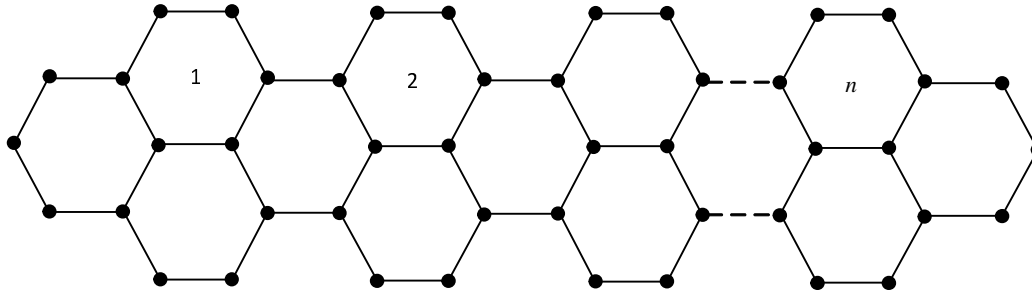

Figure 4: Benzoined system B_n of dimension n

Theorem 4. *Let G be an R_n type benzenoid system. Then $pd(G) = 3$ for $n \geq 2$.*
Proof. Since the vertices of R_n have 5 levels, we label the vertices by a_i, b_i, c_i, d_i and e_i as $1 \leq i \leq 2n$ from left to right, respectively. For illustration, labeling of the R_n type of benzenoid system is shown in Fig. 5.

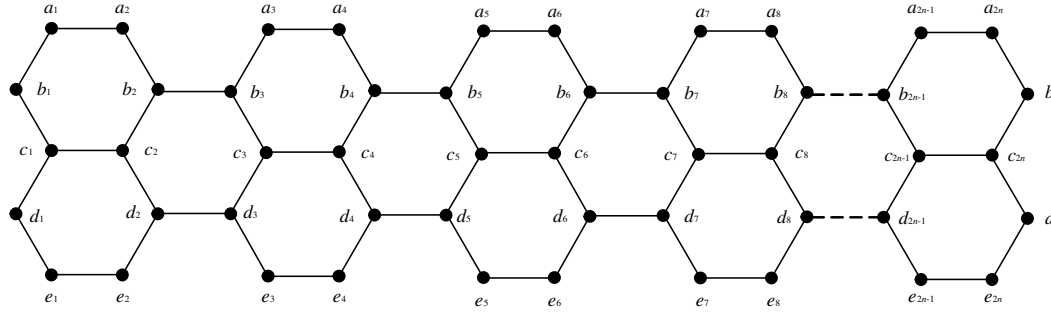

Figure 5: Labeling of the R_n benzenoid system of dimension n

Let $\Pi = \{P_1, P_2, P_3\}$ where $P_1 = \{a_i : 1 \leq i \leq 2n\}$, $P_2 = \{b_1, d_1\}$ and $P_3 = V(R_n) - \{P_1 \cup P_2\}$ be a partition of R_n. We demonstrate that Π is a resolving partition of R_n with the minimum cardinality. Each vertex in R_n with respect to Π is as follows:
For $1 \leq i \leq 2n$, each vertex's visual representation a_i of R_n with respect to Π is as follows:

$$r(a_i|\Pi) = \begin{cases} (0, i, 1) & \text{if } 1 \leq i \leq 2 \\ (2i+1, 0, 1) & \text{if } i = 3, 5, 7, \ldots, 2n-1 \\ (2i, 0, 1) & \text{if } i = 2, 4, 6, \ldots, 2n. \end{cases}$$

For $1 \leq i \leq 2n$ each vertex's visual representation b_i of R_n with respect to Π is as follows:

$$r(b_i|\Pi) = \begin{cases} (1, 0, 1) & \text{if } i = 1 \\ (1, 2i-2, 0) & \text{if } i = 3, 5, 7, \ldots, 2n-1 \\ (1, 2i-1, 0) & \text{if } i = 2, 4, 6, \ldots, 2n. \end{cases}$$

For $1 \leq i \leq 2n$ each vertex's visual representation c_i of R_n with respect to Π is as follows:

$$r(c_i|\Pi) = \begin{cases} (2, 2i-10) & \text{if } i = 1, 3, 5, \ldots, 2n-1 \\ (1, 2i-2, 0) & \text{if } i = 2, 4, 6, \ldots, 2n. \end{cases}$$

For $1 \leq i \leq 2n$ each vertex's visual representation d_i of R_n with respect to Π is as follows:

$$r(d_i|\Pi) = \begin{cases} (3, 0, 1) & \text{if } i = 1 \\ (3, 2i-2, 0) & \text{if } i = 3, 5, 7, \ldots, 2n-1 \\ (3, 2i-1, 0) & \text{if } i = 2, 4, 6, \ldots, 2n. \end{cases}$$

For $1 \leq i \leq 2n$ each vertex's visual representation e_i of R_n with respect to Π is as follows:

$$r(e_i|\Pi) = \begin{cases} (4, 2i-10) & \text{if } i = 1, 3, 5, \ldots, 2n-1 \\ (4, 2i-2, 0) & \text{if } i = 2, 4, 6, \ldots, 2n. \end{cases}$$

Every pair of vertices can be distinctly identified using the partitioning set Π. Hence Π serves as a resolving partition of R_n implying that $pd(G) \leq 3$. At the same time, Theorem 3 implies that $pd(R_n) \geq 3$. Hence $pd(R_n) = 3$.

Theorem 5. *Let G be a B_n type benzenoid system. Then $pd(G) = 3$ for $n \geq 2$.*
Proof. Since the vertices of a B_n type benzoined system has 5 levels, say l_i, m_i, n_i, o_i and p_i, we can label the vertices of $\{l_i, p_i : 1 \leq i \leq 2n\}$ and $\{m_i, o_i, n_i : 1 \leq i \leq 2n+2\}$ from left to right respectively. The labeling of a B_n type benzoined system is shown in Fig. 6.

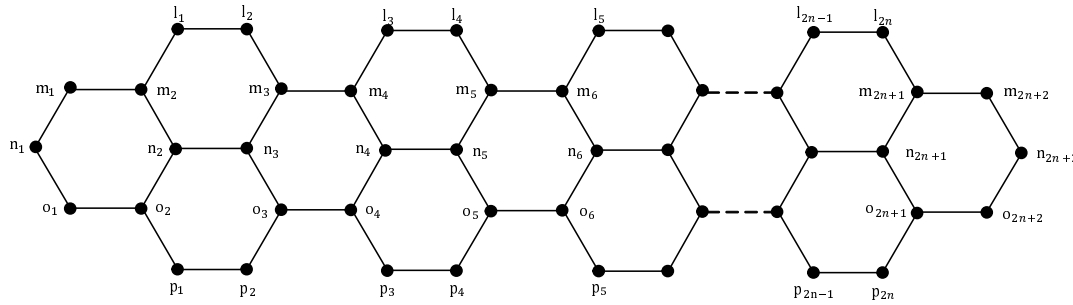

Figure 6: Labeling of the $B_n$benzenoid system of dimension n

Let $\Pi = \{P_1, P_2, P_3\}$, where $P_1 = \{n_1\}$, $P_2 = \{l_i : 1 \leq i \leq 2n\}$ and $P_3 = V(G) - \{P_1 \cup P_2\}$, be the partition set of B_n. We demonstrate that Π is a resolving partition of B_n with the minimum cardinality. Each vertex in B_n with respect to Π is as follows:

For $1 \leq i \leq 2n$, each vertex's visual representation l_i of B_n with respect to Π is as follows:

$$r(l_i|\Pi) = \begin{cases} (2i+1, 0, 1) & \text{if } i = 1, 3, 5, \ldots, 2n-1 \\ (2i, 0, 1) & \text{if } i = 2, 4, 6, \ldots, 2n. \end{cases}$$

For $1 \leq i \leq 2n+2$ each vertex's visual representation m_i of B_n with respect to Π is as follows:

$$r(m_i|\Pi) = \begin{cases} (1, i+1, 0) & \text{if } i = 1 \\ (2, i+1, 0) & \text{if } i = 2 \\ (2i-1, 1, 0) & \text{if } i = 3, 5, 7, \ldots, 2n+1 \\ (2i-2, 1, 0) & \text{if } i = 4, 6, 8, \ldots, 2n. \\ (2i-2, 2, 0) & \text{if } i = 2n+2. \end{cases}$$

For $1 \leq i \leq 2n+2$ each vertex's visual representation n_i of B_n with respect to Π is as follows:

$$r(n_i|\Pi) = \begin{cases} (0, 3, 1) & \text{if } i = 1 \\ (2i-2, 2, 0) & \text{if } i = 3, 5, 7, \ldots, 2n+1 \\ (2i-1, 2, 0) & \text{if } i = 2, 4, 6, \ldots, 2n. \\ (2i-2, 3, 0) & \text{if } i = 2n+2. \end{cases}$$

For $1 \leq i \leq 2n+2$ each vertex's visual representation o_i of B_n with respect to Π

is as follows:

$$r(o_i|\Pi) = \begin{cases} (1,4,0) & \text{if } i = 1 \\ (2,i+1,0) & \text{if } i = 2 \\ (2i-1,3,0) & \text{if } i = 3,5,7,\ldots,2n+1 \\ (2i-2,3,0) & \text{if } i = 4,6,8,\ldots,2n. \\ (2i-2,4,0) & \text{if } i = 2n+2. \end{cases}$$

For $1 \leq i \leq 2n$ each vertex's visual representation p_i of B_n with respect to Π is as follows:

$$r(p_i|\Pi) = \begin{cases} (2i+1,4,0) & \text{if } i = 1,3,5,\ldots,2n-1 \\ (2i,4,0) & \text{if } i = 2,4,6,\ldots,2n. \end{cases}$$

It follows that each vertex's representation with respect to Π is clearly different. This demonstrates that Π is a resolving partition of B_n. Therefore $pd(G) \leq 3$. Simultaneously, Theorem 3 indicates that $pd(G) \geq 3$, leading to the conclusion that $pd(G) = 3$.

3.2. Hexagonal cactus chains PC_n, MC_n and OC_n

In a hexagonal cactus chain, an internal hexagon is referred to as an ortho-hexagon, meta-hexagon or para-hexagon depending on the orientation of its cut vertices. A regular hexagonal cactus chain is a special type of graph structure consisting of a sequence of connected hexagons where each internal hexagon shares at least one vertex (called a cut-vertex) with its neighboring hexagons. In this type of cactus chains, all the internal hexagons are of the same type either ortho hexagons, metahexagons or para hexagons—depending on the position of their cut-vertices. When all hexagons are para-hexagons, the structure forms a para-chain characterized by another distinct vertex positioning. If all internal hexagons are ortho-hexagons, the chain is called an ortho-chain. These are aligned in a standard orientation. In a meta-chain, all hexagons are metahexagons where the cut-vertices are positioned differently compared to the ortho version.

Each of these chains can be represented by its length n which indicates the number of hexagons in the chain. The notations for these are typically given as OC_n for ortho hexagonal chain, MC_n for meta hexagonal chain, and PC_n for para hexagonal chain. Regular hexagonal cactus chains are often studied in graph theory for their unique structural properties and their applications in chemistry particularly in modeling molecular structures like hydrocarbons. For illustration, the para, meta and ortho hexagonal chains of dimension n are shown in Fig. 7.

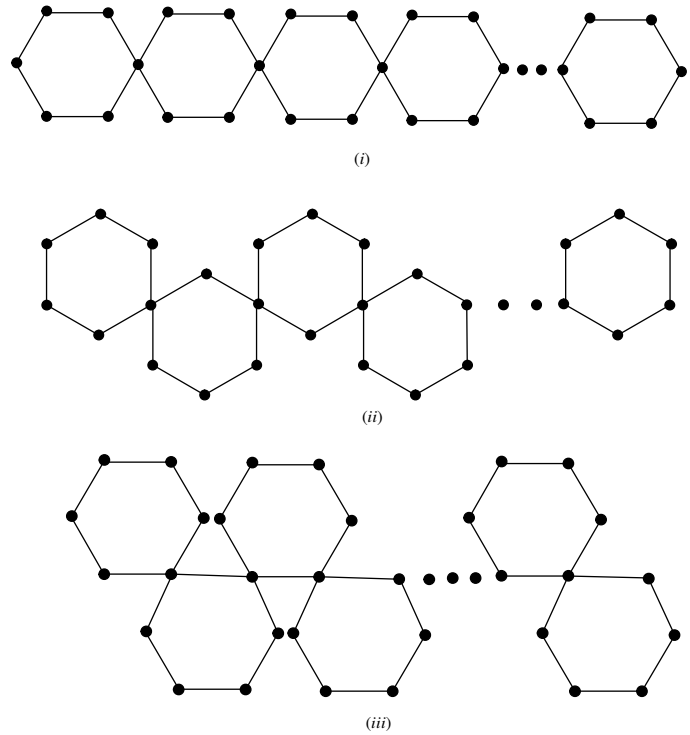

Figure 7: (i) Para hexagonal chain (ii) Meta hexagonal chain (iii) Ortho hexagonal chain with dimensions n

Theorem 6. *Let G be a para hexagonal chain (PC_n) of dimension n, $n \geq 1$. Then $pd(G) = 3$.*

Proof. Since the vertices of a para hexagonal chain (PC_n) has 3 levels say x_i, y_i and z_i, we label the vertices of x_i z_i as $1 \leq i \leq 2n$ and y_i as $1 \leq i \leq n+1$ from left to right, respectively. For illustration, labeling of a para benzene chain is shown in Fig. 8.

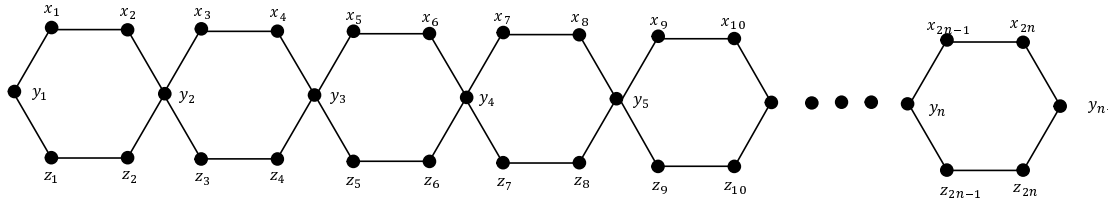

Figure 8: Labeling of the para benzene chain (PC_n) of dimension n

The partition dimension of the para-hexagonal chain graph is 3. To illustrate this, we selected a partition resolving set of cardinality 3, given by $\Pi = \{P_1, P_2, P_3\}$

where $P_1 = \{y_1\}$, $P_2 = \{x_i; 1 \leq i \leq 2n\}$ and $P_3 = V(PC_n) - \{P_1 \cup P_2\}$ be the partition resolving set of PC_n. To demonstrate the truth of this statement, we have included representations showing that each vertex of the para-hexagonal chain is distinct with respect to Π.

For $1 \leq i \leq 2n$, each vertex's visual representation x_i of PC_n with respect to Π is as follows:

$$r(x_i|\Pi) = \begin{cases} (\frac{3i-1}{2}, 0, 1) & \text{if } i = 1, 3, 5, \ldots, 2n-1 \\ (\frac{3i-2}{2}, 0, 1) & \text{if } i = 2, 4, 6, \ldots, 2n. \end{cases}$$

For $1 \leq i \leq 2n$, each vertex's visual representation y_i of PC_n with respect to Π is as follows:

$$r(y_i|\Pi) = \begin{cases} (0, 1, 1) & \text{if } i = 1 \\ (3i-3, 1, 0) & \text{if } i = 2, 3, 4, \ldots, n+1. \end{cases}$$

For $1 \leq i \leq 2n$, each vertex's visual representation z_i of PC_n with respect to Π is as follows:

$$r(z_i|\Pi) = \begin{cases} (\frac{3i-1}{2}, 2, 0) & \text{if } i = 1, 3, 5, \ldots, 2n-1 \\ (\frac{3i-2}{2}, 2, 0) & \text{if } i = 2, 4, 6, \ldots, 2n. \end{cases}$$

Every node in the para hexagonal chain's graph at the specified locations is distinct and satisfies the partition resolving set's requirements. This proved the correctness of the partition resolving number. Hence, proved that $pd(PC_n) \leq 3$. To verify this statement, we must first demonstrate that $pd(PC_n) \geq 3$. Then, assuming $pd(PC_n) = 2$ for contradiction, we find that this assumption is incorrect, as it applies only to path graphs. Hence, proved that $pd(PC_n) = 3$.

Theorem 7. *Let G be a meta hexagonal chain (MC_n) of dimension n, $n \geq 1$. Then $pd(G) = 3$.*

Proof. Since the vertices of meta hexagonal chain (MC_n) has 7 levels, we label the vertices of q_i, s_i, w_i, u_i as $1 \leq i \leq \frac{n}{2}$, r_i and v_i as $1 \leq i \leq n$ and t_i as $1 \leq i \leq n+1$, from left to right, respectively. For illustration, labeling of a meta benzene chain is shown in Fig. 9.

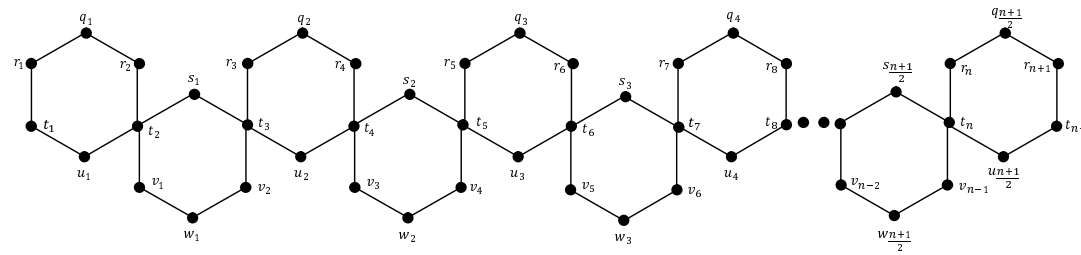

Figure 9: Labeling of the meta benzene chain MC_n of dimension n

The partition dimension of the meta-hexagonal chain graph is 3. To illustrate this, we selected a partition resolving set of cardinality 3 given by $\Pi = \{P_1, P_2, P_3\}$ where $P_1 = \{r_1, t_1\}$, $P_2 = \{s_i \cup w_i \cup u_i; 1 \leq i \leq \frac{n}{2}\}$ and $P_3 = V(MC_n) - \{P_1 \cup P_2\}$ be a resolving partition set of MC_n. We have given representations that show that every vertex of the meta-hexagonal chain is distinct with regard to Π in order to prove the truth of this statement.

For $1 \leq i \leq \frac{n}{2}$, each vertex's visual representation q_i of MC_n with respect to Π is as follows:

$$r(q_i|\Pi) = \begin{cases} (1,3,0) & \text{if } i = 1 \\ (4i-2,3,0) & \text{if } 2 \leq i \leq \frac{n}{2}. \end{cases}$$

For $1 \leq i \leq n$, each vertex's visual representation r_i of MC_n with respect to Π is as follows:

$$r(r_i|\Pi) = \begin{cases} (0,2,1) & \text{if } i = 1 \\ (2,2,0) & \text{if } i = 2 \\ (2i-2,2,0) & \text{if } i = 3,4,5,\ldots,n. \end{cases}$$

For $1 \leq i \leq \frac{n}{2}$, each vertex's visual representation s_i of MC_n with respect to Π is as $r(s_i|\Pi) = (4i-1,0,1)$.

For $1 \leq i \leq n+1$, each vertex's visual representation t_i of MC_n with respect to Π is as follows:

$$r(t_i|\Pi) = \begin{cases} (0,1,1) & \text{if } i = 1 \\ (2i-2,1,0) & \text{if } 2 \leq i \leq n+1. \end{cases}$$

For $1 \leq i \leq \frac{n}{2}$, each vertex's visual representation u_i of MC_n with respect to Π is as $r(u_i|\Pi) = (4i-3,0,2)$.

For $1 \leq i \leq n$, each vertex's visual representation v_i of MC_n with respect to Π is as $r(v_i|\Pi) = (2i+1,1,0)$.

For $1 \leq i \leq \frac{n}{2}$, each vertex's visual representation w_i of MC_n with respect to Π is as $r(w_i|\Pi) = (4i,0,1)$.

Every node in the meta hexagonal chain's graph at the specified locations is distinct and satisfies the partition resolving set's requirements. This proved the correctness of the partition resolving number. Hence, we already proved that $pd(MC_n) \leq 3$. To verify the statement, we must demonstrate that $pd(MC_n) \geq 3$. Then, assuming $pd(MC_n) = 2$ for contradiction, we find that this assumption is incorrect, as it applies only to path graphs. Hence, we have proved that $pd(MC_n) = 3$.

Theorem 8. *Let G be a ortho hexagoanl chain (OC_n) of dimension n, $n \geq 1$. Then $pd(G) = 3$.*

Proof. The vertices of ortho hexagonal chain (OC_n) has 5 levels say k_i, l_i, m_i, n_i and o_i. When n is odd, we label the vertices of k_i, l_i, m_i as $1 \leq i \leq n+1$ and n_i

and o_i as $1 \leq i \leq n-1$, while when n is even, the labeling of the vertices of k_i, i_i, n_i and o_i as $1 \leq i \leq n$ and m_i as $1 \leq i \leq n+1$ from left to right, respectively. For illustration, labeling of an ortho benzene chain OC_n are shown in Fig. 10.

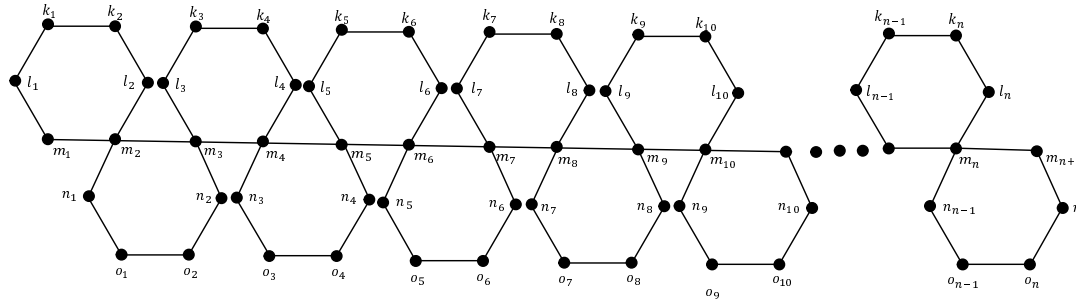

Figure 10: Labeling of the ortho benzene chain (OC_n) of dimension n

The partition dimension of the ortho hexagonal chain graph is 3. To illustrate this, we selected a partition resolving set of cardinality 3, given by $\Pi = \{P_1, P_2, P_3\}$ where $P_1 = \{o_1\}$, $P_2 = \{k_i; for\ n \text{ is odd } 1 \leq i \leq n+1, \text{ for } n \text{ is even } 1 \leq i \leq n\}$ and $P_3 = V(OC_n) - \{P_1 \cup P_2\}$ be the resolving partition set of OC_n. To confirm the validity of this statement, we have provided representations demonstrating that each vertex of the ortho hexagonal chain is uniquely distinguished with respect to Π.

For n is odd, $1 \leq i \leq n+1$ and for n is even, $1 \leq i \leq n$. Each vertex's visual representation k_i of OC_n with respect to Π is as $r(k_i|\Pi) = (i+1, 0, 1)$.

For n is odd, $1 \leq i \leq n+1$ and for n is even, $1 \leq i \leq n$. Each vertex's visual representation l_i of OC_n with respect to Π is as $r(l_i|\Pi) = (i, 1, 0)$.

For $1 \leq i \leq n+1$, each vertex's visual representation m_i of OC_n with respect to Π is as follows:

$$r(m_i|\Pi) = \begin{cases} (i-1, 2, 1) & \text{if } i = 1 \\ (i-1, 2, 0) & \text{if } 2 \leq i \leq n+1 \end{cases}$$

For n is odd, $1 \leq i \leq n-1$ and for n is even, $1 \leq i \leq n$ each vertex's visual representation n_i of OC_n with respect to Π is as $r(n_i|\Pi) = (i+1, 3, 0)$.

For n is odd, $1 \leq i \leq n-1$ and for n is even, $1 \leq i \leq n$. Each vertex's visual representation o_i of OC_n with respect to Π is as $r(o_i|\Pi) = (i+2, 4, 0)$.

Every node in the ortho hexagonal chain's graph at the specified locations is distinct and satisfies the partition resolving set's requirements. This proved the correctness of the partition resolving number. Hence, we have proved that $pd(OC_n) \leq 3$. To verify the claimed statement, we must demonstrate that $pd(OC_n) \geq 3$ as before.

Then, again assuming that $pd(OC_n) = 2$,
we get a contradiction.

4. Theoretical importance and practical use of constant partition dimension in hexagonal cactus chains and certain pericondensed benzenoid systems

The study of the constant partition dimension of hexagonal cactus chains and certain pericondensed benzenoid systems is a specialized and significant area of graph theory. Understanding this topic requires exploring both its theoretical foundations and practical implications to fully understanding its importance and potential applications.

The theoretical importance of pericondensed benzenoid systems and hexagonal cactus chains in relation to partition dimension concepts lie in their structural complexity and symmetry. These systems provide a rich framework for studying partition dimension, as their unique topologies allow for the exploration of how vertices can be optimally partitioned into subsets to resolve the graph. This contributes to a deeper understanding of graph identification, optimization problems and applications in chemical graph theory particularly in modeling molecular structures like hydrocarbons and cyclic compounds.

The practical applications of partition dimension related to pericondensed benzenoid systems are primarily found in chemical graph theory, particularly in the modeling and analysis of molecular structures. In these systems, partition dimension helps in uniquely identifying the position of atoms or functional groups within a molecule, which is crucial for understanding chemical reactivity, stability, and synthesis pathways. It can also be applied in the design of chemical sensors and molecular recognition systems where precise identification of molecular components is necessary for detecting chemical properties or designing drugs. Additionally, partition dimension can assist in optimizing network designs in computational chemistry for efficient data handling and molecular simulations.

The most practical application of partition dimension related to hexagonal cactus chains is primarily in network optimization and chemical structure modeling. In these systems, partition dimension aids in uniquely identifying vertices which is valuable for efficient routing, monitoring, and navigation in network structures such as sensor networks or communication systems. In chemical graph theory, hexagonal cactus chains model molecular frameworks like benzenoid hydrocarbons, and partition dimension helps in analyzing the molecular structure for chemical stability, reactivity and synthesis. This concept also finds use in designing algorithms for robotics navigation and chemical recognition systems.

Studies on hexagonal cactus chains and certain pericondensed benzenoid sys-

tems' constant partition dimension add to the theoretical foundations of graph theory and its practical applications across a range of fields. It may lead to developments in data analysis, network optimization, algorithm development and other fields where graph theory is crucial.

The partition dimension provides a measure of how efficiently the vertices of a molecular graph can be uniquely distinguished using a small number of vertex partitions. For the studied benzenoid systems and para-, meta-, and ortho-hexagonal cactus chains, a partition dimension of 3 implies that all atomic positions in these molecular frameworks can be uniquely identified using three resolving subsets. This reflects a low level of structural ambiguity and indicates that these chemical graphs possess a high degree of symmetry and regularity. From a chemical perspective, such unique identification of vertices is relevant to distinguishing atomic environments, analyzing isomeric structures, and simplifying computational models used in molecular simulations and structure–property analysis.

5. Concluding remarks

In this study, we computed the partition dimension for two special benzenoid systems, para, meta and ortho hexagonal cactus chains is 3. By examining the distinct structural features of each system, we demonstrated that the partition dimension varies according to the arrangement and connectivity of the hexagonal units within the chains. For para-hexagonal, meta-hexagonal and ortho-hexagonal systems, the partition dimension was found to be influenced by both the symmetry and the specific geometric relationships between adjacent hexagons. Our analysis showed that the partition dimension provides a precise metric to distinguish vertices within these complex molecular graphs. For each type of hexagonal cactus chain, we established a minimal resolving partition set that could uniquely identify every vertex. This distinction highlights the critical role that the structural configuration plays in determining the partition dimension for each chain.

Acknowledgement

This research was supported by the University of Phayao and the Thailand Science Research and Innovation Fund (Fundamental Fund 2026, Grant No. 2268/2568).

References

[1] Ahmad A., Koam A. N. A., Siddiqui M. H. F., and Azeem M., Resolvability of the starphene structure and applications in electronics, Ain Shams Engineering journal, 13 (2021), 1-9.

[2] Ahmad A., Baca M., and Sultan M. S., Computing the metric dimension of

kayak paddle graph and cycles with chord, Proyecciones Journal of Mathematics, 39 (2020), 287–300.

[3] Alishahi M. and Shalmaee S. H., On the edge eccentric and modified edge eccentric connectivity indices of linear benzenoid chains and double hexagonal chains, Journal of molecular structure, 1204 (2020), 127446.

[4] Chartrand G., Salehi E., and Zhang P., The partition dimension of a graph, Aequationes Mathematicae, 59 (2000), 45-54.

[5] Chartrand G., Eroh L., Johnson M. A. O., and Ortrud R., Resolvability in graphs and the metric dimension of a graph, Discrete Applied Mathematics, 105 (2000), 99–113.

[6] Cruz R., Marín C. A. and Rada J., Computing the hosoya index of catacondensed hexagonal systems, MATCH- Communication in Mathematical and in Computer Chemistry, 77 (2017), 749-764.

[7] Doslic T., and Maloy F., Chain hexagonal cacti matchings and independent sets, Discrete Mathematics, 310 (2010), 1676-1690.

[8] Garey M. R., and Johnson D. S., Computers and intractability, A Guide to the Theory of NP-Completeness, Freeman, New York, 1979.

[9] Gultekin I. and Sahin B., Some relations betweenk Kekulé structure and morgan-voyce polynomials, Iranian Journal of Mathematical Chemistry, 8 (2) (2017), 221-229.

[10] Gutman I., Kolakovic N., Graovac A. and Babic D., A method for calculation of the hosoya index of polymers, Studies in Physical and Theoretical Chemistry, 63 (1989), 141-154.

[11] Gutman I., Extremal hexagonal chains, Journal of Mathematical Chemistry, 12 (1993), 197-210.

[12] Harary F., and Melter R. A., On the metric dimension of a graph, Ars Combinatoria, 2 (1976), 191–195.

[13] Hussain Z., Munir M., Choudhary M., and Kang S. M., Computing metric dimension and metric basis of the 2D lattice of alpha-boron nanotubes, Symmetry, 10 (2018), 300.

[14] Javaid I., and Shokat S., On the partition dimension of some wheel related graphs, Journal of Prime Research in Mathematics, 4 (2008), 154–164.

[15] Johnson M. A., Structure-activity maps for visualizing the graph variables arising in drug design Journal of bio pharmaceutical Statistics, 3 (1993), 203–236.

[16] Khabyah A. A., Koam A. N. A., and Ahmad A., Partition resolvability of nanosheet and nanotube derived from octagonal grid, Journal of Mathematics, (2024), 6222086.

[17] Khuller S., Raghavachari B., and Rosenfeld A., Landmarks in graphs, Discrete Applied Mathematics, 70 (1996), 217–229.

[18] Koam A. N. A., Ahmad A., Azeem M., and Nadeem M. F., Bounds on the partition dimension of one pentagonal carbon nanocone structure, Arabian Journal of Chemistry 15 (2022), 1-6.

[19] Oz M. S. and Cangul I. N., Computing the hosoya and the merrifield-simmons indices of two special benzenoid systems, Iranian Journal of Mathematical Chemistry, 12 (2021), 161-174.

[20] Ramdhani V., and Rahmi F., The partition dimension of a path graph, Sainstek Jurnal Sains dan Teknologi, 13 (2021), 66–72.

[21] Sadeghieh A., Alikhani S., Ghanbari N., and Khalaf A. J. M., Hosoya polynomial of some cactus chains, Cogent Mathematics, 4 (2017), 1-7.

[22] Sharma S. K., Singh M., and Bhat B. K., Vertex-edge partition resolvability for certain carbon nanocones, Polycyclic Aromatic Compounds, 44 (2023), 1745–1759.

[23] Slater P. J., Leaves of trees, Proceeding of the 6th southeastern conference combinatorics, Graph Theory, and Computing, 14 (1975), 549–559.

[24] Tomescu I., Discrepancies between metric dimension and partition dimension of a connected graph, Discrete Mathematics, 308 (2008), 5026–5031.

[25] Turaci T. and Durgut R., On topological properties of some molecular cactus chain networks and their subdivisions by using line operator, Journal of Discrete Mathematical Sciences and Cryptography, 25 (2021), 2597-2614.

[26] Wang W., Jin Y., Mu Y., Zhang M., and Du J., A novel tubular structure with negative poisson's ratio based on gyroid-type triply periodic minimal surfaces, Virtual and Physical Prototyping, 18 (2023), 1-17.

[27] Yarahmadi Z., Doslic T., and Moradi S., Chain hexagonal cacti extremal with respect to the eccentric connectivity index, Iranian Journal of Mathematical Chemistry, 4 (2016), 123-136.